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Secrets of the metric in $\mathcal{N}=4$ and $\mathcal{N}=2^*$ geometries

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Abstract: The metric of the gravity dual of a field theory should contain precisely the same information as the field theory. We discuss this connection in the $\mathcal{N}=4$ theory where a scalar vev may be introduced at the level of 5d supergravity and the solutions lifted to 10d. We stress the role of brane probing in finding the coordinates appropriate to the field theory. In these coordinates the metric parametrizes the gauge invariant operators of the field theory and either side of the duality is uniquely determined by the other. We follow this same chain of computations for the 10d lift of the $\mathcal{N}=2^*$ geometry of Pilch and Warner. The brane probe of the metric reveals the 2d moduli space and the functional form of the gauge coupling. In the coordinates appropriate to the field theory the metric on moduli space takes a very simple form and one can read off the gravity predictions for operators in the field theory. Surprisingly there is logarithmic renormalization even in the far UV where the field theory reverts to $\mathcal{N}=4$ super Yang-Mills. We extend the analysis of Buchel et al to find the D3-brane source distribution that generates the supergravity prediction for the gauge coupling for the whole class of solutions corresponding to different points on moduli space. This distribution does not account for the logarithmic behaviour in the rest of the metric though. We discuss possible resolutions of the discrepancy.

Keywords: AdS-CFT Correspondance, Brane Dynamics in Gauge Theories, Renormalization Regularization and Renormalons.

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Contents

1. Introduction

The AdS/CFT correspondence [1] has provided the first example of a fascinating duality between a strongly coupled gauge theory and a weakly coupled gravity background. It has immediately been of interest to extend such dualities to other gauge theories and gravity backgrounds to understand how generic such a duality is. A number of techniques have been used to push forward these explorations; finite temperature may be included by compactification of the time direction [3, 4], relevant deformations can be included by switching on appropriate supergravity fields that act as sources in the $\mathcal{N} = 4$ theory [5]–[20], and new D brane structures with different world volume theories and their near horizon geometries may be constructed [21]–[27]. In this paper we want to make a deeper investigation of the anatomy of some of these dualities. In principle two theories which are dual should simply be reparametrizations of the same "solution". Thus if we know the complete solution to some field theory the corresponding gravity dual should be uniquely determined. Understanding how this encoding occurs in some simple theories will hopefully lead to new tools for constructing a wider class of dualities. In this paper we take some steps in this direction.

In particular we will begin by revisiting the gravity duals of $\mathcal{N} = 4$ super Yang Mills on moduli space. The field theory is well understood and the gravity duals have been deduced both from D3-brane constructions [21, 12] and from deformed 5d supergravity solutions [11] lifted to 10d solutions [16]. The latter solutions have been shown to match the former. Here we will approach this connection from the field theory side. Starting with the 10d lift supergravity solution we will show that brane probing the solution provides a simple tool for determining the unique coordinates it should be written in to make the gauge theory correspondence manifest. The Dirac-Born-Infeld action of the probe provides a swift link between the gravity metric and the dimensionality of the field theory moduli space and the functional dependence of the gauge coupling on that moduli space. In the special coordinates there is a manifest prescription for the encoding of the field theory operators in the metric. The 5d supergravity solutions only describe a subset of the possible moduli space but using the prescription the full set of 10d supergravity solutions needed to describe the full moduli space may be deduced. These metrics are indeed solutions of the 10d supergravity equations of motion.

To test whether the encoding prescription is generic we move these ideas across to the gravity dual of the $\mathcal{N} = 2^*$ theory (the N=4 theory with a mass term that breaks supersymmetry to $\mathcal{N} = 2$ in the IR) which has more interesting RG flow properties. Solutions produced by including relevant deformations in the 5d supergravity theory [14, 17, 18] have been lifted to 10d by Pilch and Warner [16] and also in [17]. The conection to the gauge theory of this set of solutions is far from apparent after the lift. The use of a brane probe to uncover the links was made in [19] and [20]. The metric indeed describes the expected 2d moduli space of the field theory. The gauge coupling function on the moduli space is also revealed and, when the solution is placed in appropriately $\mathcal{N} = 2$ coordinates [20], matches to field theory expectations. The set of solutions describe different points on moduli space with one corresponding to a singular point on moduli space where in the IR the gauge coupling diverges. This solution is of interest because it provides an example of the enhançon mechanism [22] (there are points in the space where a probe's tension falls to zero). We write the metric on moduli space in the coordinates applicable to the field theory where it takes the form of a single function as in the $\mathcal{N} = 4$ metrics multiplied by the gauge coupling function. It is natural to interpret the outstanding function according to the same prescription as in the $\mathcal{N} = 4$ solution and read off field theory operators. In the field theory the gauge coupling encodes the only renormalization group (RG) flow [32] whilst the supergravity solution appears to describe additional renormalization of the scalar operators. In addition in the far UV the solution does not return to the $\mathcal{N} = 4$ form but contains logarithmic renormalization. To highlight the discrepancy we follow the prescription in [20] for deducing the D3-brane distribution from the expected field theory gauge coupling, as a function of position on moduli space, and the supergravity form for the coupling. We thus deduce the distribution for all the 5d supergravity lifts and can then calculate the expected scalar operators which again do not match with the function in the metric. Presumably there is some discrepancy in the prescription in this more complicated theory we have not yet discovered, nevertheless, we believe the approach to be valuable and the discrepancies an important starting point for future progress.

2. The gravity dual of $\mathcal{N} = 4$ on moduli space

We begin by studying gravity solutions describing $\mathcal{N} = 4$ super Yang-Mills (SYM) theory on moduli space resulting from 5d supergravity [11]. We wish to study the

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gauge theory in the presence of a non-zero vacuum expectation value (vev) for the scalar operator $\mathrm{tr}\,\phi^2$. Since there are 6 real scalars this operator is a 6x6 matrix with the symmetric traceless entries transforming as the 20 of the global $\mathrm{SU}(4)_R$ symmetry of the theory. In the 5d truncation of IIB supergravity on $AdS_5 \times S^5$ [28, 29] the lightest state is a scalar, α, in the 20 that acts as the source for $\mathrm{tr}\,\phi^2$ in the AdS/CFT correspondence [2, 3]. One may look for solutions of the 5d supergravity equations of motion with non-zero α and interpret them as gravity duals of the $\mathcal{N}=4$ theory with a scalar vev switched on. In fact, considerable work is needed to arrive at the equations of motion since the scalars live in the coset $E_6/\,\mathrm{USp}(8)$, the subtleties of which are discussed in [11]. We shall present the final results only here.

As an example let us consider the case of switching on $\mathrm{tr}\,\phi^2 = \mathrm{diag}(1,1,1,1-2,-2)$. The appropriate supergravity scalar has been identified in [11]. In the supergravity theory the metric is dynamical and the scalar vev cannot be considered in isolation. We parametrize the metric as

$$ds^2 = e^{2A(r)} dx_{//}^2 - dr^2\,, \tag{2.1}$$

where $x_{//}$ describe four dimensional Minkowski space slices through the deformed AdS space, r is the radial direction, and in the AdS limit $A(r) = r/L$ with L the radius of the AdS space. The resulting supersymmetric equations of motion (for which the fermionic shifts vanish) are first order (where $\rho = e^{\alpha}$),

$$\frac{\partial \rho}{\partial r} = \frac{1}{3L}\left(\frac{1}{\rho} - \rho^5\right), \qquad \frac{\partial A}{\partial r} = \frac{2}{3L}\left(\frac{1}{\rho^2} + \frac{\rho^4}{2}\right). \tag{2.2}$$

These equations may be solved in the $\rho - A$ plane since

$$\frac{\partial \rho}{\partial A} = \frac{1}{2}\left(\frac{\rho - \rho^7}{1+\rho^6/2}\right), \tag{2.3}$$

with solution

$$e^{2A} = \frac{l^2}{L^2}\frac{\rho^4}{\rho^6 - 1}\,, \tag{2.4}$$

with l^2/L^2 a constant of integration. At this level the connection to the dual gauge theory is somewhat opaque.

Remarkably the solution has been lifted back to a $D = 10$ solution [11, 16] which takes the form

$$ds^2 = \frac{X^{1/2}}{\rho} e^{2A(r)} dx_{//}^2 - \frac{X^{1/2}}{\rho}\left(dr^2 + \frac{L^2}{\rho^2}\left[d\theta^2 + \frac{\sin^2\theta}{X} d\phi^2 + \frac{\rho^6 \cos^2\theta}{X} d\Omega_3^2\right]\right), \tag{2.5}$$

where $d\Omega_3^2$ is the metric on a 3-sphere and

$$X \equiv \cos^2\theta + \rho^6 \sin^2\theta\,. \tag{2.6}$$

For consistency there must also be a non-zero C_4 potential of the form

$$C_4 = \frac{e^{4A}X}{g_s\rho^2} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \,. \tag{2.7}$$

Again any relation to a dual theory is well hidden. In fact in [11] this metric was determined to be equivalent to the near horizon limit of a multi-centre solution around a D3-brane distribution. We wish to make the need for the transformation to these coordinates clear within the context of the duality. There are many coordinate redefinitions one could make and only a single set of coordinates can manifestly display the field theory duality. A tool is needed to find these coordinates and the appropriate choice for that tool, as we will see, is brane probing.

Brane probing [30] is most transparent in the orginal D3-brane construction for the AdS/CFT correspondence. Here there is a stack of N D3-branes at the origin with the $\mathcal{N} = 4$ SYM as their world volume theory and $AdS_5 \times S^5$ as their near horizon geometry. If we imagine moving a single D3-brane from the stack and moving it in the space then, to first order, it will not effect the background metric. From the world volume field theory point of view, by separating a D3-brane we have introduced an adjoint scalar vev breaking $\mathrm{SU}(N) \to \mathrm{U}(1) \times \mathrm{SU}(N-1)$. The magic of D-branes is that the scalar fields' vevs in the field theory are precisely identified with the position of the D3-brane in the surrounding spacetime. This is expressed by the Dirac Born Infeld (DBI) action for a D3-brane,

$$S_{\rm probe} = -\tau_3 \int_{\mathcal{M}_4} d^4x \, \det \left[G^{(E)}_{ab} + 2\pi\alpha' e^{-\Phi/2} F_{ab} \right]^{1/2} + \mu_3 \int_{\mathcal{M}_4} C_4 \,, \tag{2.8}$$

where G_{ab} is the pull back of the spacetime metric, F^{ab} the gauge field on the probe's surface, Φ the dilaton (which is a constant in this solution) and $\tau_3 = \mu_3/g_s$. Thus the DBI action allows us to translate the background metric to a potential for the scalar fields in the field theory. It is easy to identify the dimension of the field theory moduli space implied by the metric from where the DBI potential vanishes. In addition since the U(1) theory lives on the probe's surface and is a non-interacting theory (photons do not self interact and there is only adjoint matter which for a U(1) is chargeless), its coupling is that of the $\mathrm{SU}(N)$ theory at the scale of the breaking vev. The probe also therefore lets us determine the functional form of the coupling on moduli space.

We proceed to brane probe the 10d metric above by substituting (2.5)–(2.7) in (2.8). Allowing the brane to move slowly and concentrating on the scalar sector, we find the DBI action corresponds to the field theory,

$$S = -\frac{\mu_3}{2g_s} \int_{\mathcal{M}_4} d^4x \left[\frac{Xe^{2A}}{\rho^2} \dot{r}^2 + \frac{L^2 e^{2A}}{\rho^4} \left(X\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 + \rho^6 \cos^2\theta\, \dot{\Omega}_3^2 \right) \right] . \tag{2.9}$$

The immediate result is that we see there is no potential against motion of the probe in the full 6 dimensional transverse space corresponding in the field theory to the scalars having a 6d moduli space. This matches with our expectations for the $\mathcal{N}=4$ SYM theory where the six scalars have a potential of the form $\mathrm{tr}[\phi,\phi]^2$ and so, taking commuting vevs, the six scalars may take arbitrary values.

The kinetic terms should be interpreted as the kinetic terms of the field theory scalars which in the $\mathcal{N}=4$ theory are given by $(1/8\pi)\,\mathrm{Im}(\tau\Phi^\dagger\Phi)|_D$ (in $\mathcal{N}=1$ notation). The coefficient of the kinetic terms are therefore the gauge coupling which is known to be conformal in the $\mathcal{N}=4$ theory. We should expect the metric that the probe sees on moduli space to be flat which it manifestly isn't in (2.9). This is our hint as to the coordinate change we should make in order to pass to those coordinates where the duality is manifest. Forcing this relation we find a change of coordinates that makes the probe metric flat

$$(r,\theta)\to(u,\alpha) \tag{2.10}$$

such that

$$u^2\cos^2\alpha = L^2 e^{2A}\rho^2\cos^2\theta\,, \qquad u^2\sin^2\alpha = L^2\frac{e^{2A}}{\rho^4}\sin^2\theta\,. \tag{2.11}$$

A small calculation shows that the metric in these coordinates takes the form

$$S=-\frac{\mu_3}{2g_s}\int_{\mathcal{M}_4} d^4x\left[\dot{u}^2+u^2\left(\dot{\alpha}^2+\sin^2\alpha\,\dot{u}^2+\cos^2\alpha\,\dot{\Omega}_3^2\right)\right]. \tag{2.12}$$

This is a unique choice of coordinates and if the duality is to be manifest it must be in these coordinates where the coupling is seen to have the correct conformal property. It is therefore interesting to write the full metric in these coordinates

$$ds^2=\left(\frac{\rho^2}{Xe^{4A}}\right)^{-1/2}dx_{//}^2+\left(\frac{\rho^2}{Xe^{4A}}\right)^{1/2}\sum_{i=1}^{6}(du_i)^2\,. \tag{2.13}$$

This is of the familiar form,

$$ds^2=H^{-1/2}dx_{//}^2+H^{1/2}\sum_{i=1}^{6}du_i^2\,,\qquad C_4=\frac{1}{Hg_s}dx^0\wedge dx^1\wedge dx^2\wedge dx^3\,. \tag{2.14}$$

From the coordinate transformations (2.11) and using (2.4), we can obtain an explicit expression for ρ in terms of (u,α)

$$\frac{u^2}{l^2}\sin^2\alpha\,\rho^{12}+\left(\frac{u^2}{l^2}\cos^2\alpha-\frac{u^2}{l^2}\sin^2\alpha-1\right)\rho^6-\frac{u^2}{l^2}\cos^2\alpha=0\,. \tag{2.15}$$

In [11] it was shown that in these coordinates $H(u)$ can be written as a multi-centre solution with a D3 density, σ,

$$H(u)=\int d^6x\,\sigma(x)\,\frac{L^4}{|\vec{u}-\vec{x}|^4}\,. \tag{2.16}$$

In this case the density is a 2 dimensional disk of uniform density in the $\theta = \pi/2$ plane

$$\sigma(x) = \frac{1}{\pi l^2}\theta(l^2 - x^2)\,. \tag{2.17}$$

We wish to make the connection to the field theory and instead consider the large u limit of (2.15) from which we obtain

$$\rho^6 = 1 + \frac{l^2}{u^2} + \left(\frac{l^2}{u^2}\right)^2 (1 - \sin^2\alpha) + \left(\frac{l^2}{u^2}\right)^3 (1 - 3\sin^2\alpha + 2\sin^4\alpha) + \mathcal{O}\left(\frac{l^8}{u^8}\right). \tag{2.18}$$

and hence from (2.13)

$$H(u) = \frac{L^4}{u^4}\left(1 + \frac{l^2}{u^2}(3\sin^2\alpha - 1) + \frac{l^4}{u^4}(1 - 8\sin^2\alpha + 10\sin^4\alpha)\right) + \mathcal{O}\left(\frac{L^4 l^6}{u^{10}}\right). \tag{2.19}$$

In this form it is possible for us to identify field theory operators [21]. The radial coordinate u has the scaling dimension of mass [3] so in each term in the expansion we can assign a scaling dimension to the coefficient. Further each term in the expansion is associated with a unique spherical harmonic;[1] the angular function in the $1/u^6$ term is the spherical harmonic in the 20 of $SU(4)_R$, that in the $1/u^8$ term the harmonic in the 50 and so forth. Note that by using the orthonormality of the spherical harmonics it is easy to show that each harmonic occurs only in a single term in the expansion. We can therefore identify the nth coefficient as having the dimension and symmetry properties of the field theory operator $\mathrm{tr}\,\phi^n$ and further that the operator is not renormalized since there is no further function of u associated with the operator. Thus these solutions suggest the general form

$$H(u) = \frac{L^4}{u^4}\left(1 + \sum_n \frac{\mathrm{tr}\,\phi^n}{u^n} Y_n\right), \tag{2.20}$$

where Y_n is the spherical harmonic obtained from the product of n 6 dimensional reps.

It is worth noting that at the level of the 5d supergravity theory we introduced only a vev for the dimension 2 operator $\mathrm{tr}\,\phi^2$ yet after the lift to 10d the solution was forced to possess vevs for higher dimension operators. If we returned to 5d the truncation would again remove these operators. The 5d supergravity metric gives specific relations between the operators as is explicit in (2.19) whilst in the field theory they are expected to be arbitrary reflecting the 6 dimensional moduli space. One may therefore try substituting the expansion with arbitrary coefficients into the

[1] The spherical harmonics may be found by writing the 6 dimensional representation as a unit vector in the transverse space and then finding the symmetric traceless products $6 \times 6 = 20 + \cdots$, $6 \times 6 \times 6 \times 6 = 50 + \cdots$, etc.

supergravity field equations and they indeed turn out to be solutions [21]. Of course in this context this is no surprise because it is already known that the multi-centre solutions are solutions of the field equations for arbitrary D3-brane distributions. However, it is encouraging in this simplest case that one can deduce a full gravity description of the field theory from the 5d supergravity solutions. Further it is appealing that the metric is indeed seen to be a rewriting of the field theory solutions and it is of interest to see how this generalizes in theories with more complicated RG flow. In the next section we will study aspects of this generalization for the $\mathcal{N} = 2^*$ theory.

Before moving on though we wish to note the power of the brane probing technique since it in fact is capable of deriving the above solutions on its own. In the $\mathcal{N} = 4$ case if we wished to write down a metric dual to a point on moduli space we might begin by writing down an arbitrary 10d metric. If we then require the 6 dimensional moduli space and conformal coupling after a brane probe the metric is forced to take the form in (2.14). The supergravity field equations with this ansatz reduce to the transverse flat space laplacian in 6 dimensions [31],

$$\Delta_6 H(u) = 0\,. \tag{2.21}$$

Which produces the multi-centre solutions. We see again that when we know sufficient information about the field theory the supergravity dual is uniquely determined.

3. The $\mathcal{N} = 2^*$ geometry

We have seen that in the $\mathcal{N} = 4$ duality there is a simple mapping between the field theory operators and the form of the metric. It would be interesting to understand how this mapping occurs in a more complicated theory with non-trivial renormalization group flow. The theory we choose to investigate in this light is the $\mathcal{N} = 2^*$ theory where a mass term is introduced into the $\mathcal{N} = 4$ theory that leaves an $\mathcal{N} = 2$ supersymmetric theory in the IR. The 5d supergravity theory with the appropriate supergravity field deformations switched on was studied in [14, 17, 18]. Two supergravity scalars are needed, one describing the mass term and the other the possible vev for the remaining two real scalar fields. Although some connections were made between the field theory and these solutions the duality remained fairly opaque at the 5d level. A lift of this solution to 10d supergravity has again been provided [16, 17] and the summary of the solution is

$$ds^2 = \Omega^2(e^{2A}dx_{//}^2 + dr^2) + \frac{L^2\Omega^2}{\rho^2}\left(\frac{d\theta^2}{c} + \rho^6\cos^2\theta\left(\frac{\sigma_1^2}{cX_2} + \frac{\sigma_1^2+\sigma_2^2}{X_1}\right) + \frac{\sin^2\theta}{X_2}d\phi^2\right), \tag{3.1}$$

where

$$
\begin{aligned}
\Omega^2 &= \frac{(cX_1X_2)^{1/4}}{\rho}\,, & c &= \cosh 2m \\
X_1 &= \cos^2\theta + \rho^6 c\sin^2\theta\,, & X_2 &= c\cos^2\theta + \rho^6\sin^2\theta \\
C_4 &= \frac{e^{4A}X_1}{g_s\rho^2}dx^0\wedge dx^1\wedge dx^2\wedge dx^3\,.
\end{aligned}
\tag{3.2}
$$

The dilaton is non-trivial too. We write a complex scalar $\lambda = C_0 + ie^{-\Phi}$ and

$$
\lambda = i\left(\frac{1-B}{1+B}\right), \qquad B = \left(\frac{b^{1/4}-b^{-1/4}}{b^{1/4}+b^{-1/4}}\right), \qquad b = \cosh(2m)\frac{X_1}{X_2}\,. \tag{3.3}
$$

The fields m, A and $\rho = e^\alpha$ are the supergravity fields given by the 5d supergravity equations of motion

$$
\begin{aligned}
\frac{\partial\alpha}{\partial r} &= \frac{1}{3L}\left(\frac{1}{\rho^2} - \rho^4\cosh(2m)\right) \\
\frac{\partial A}{\partial r} &= \frac{2}{3L}\left(\frac{1}{\rho^2} + \frac{1}{2}\rho^4\cosh(2m)\right) \\
\frac{\partial m}{\partial r} &= -\frac{1}{2L}\rho^4\sinh(2m)\,,
\end{aligned}
\tag{3.4}
$$

which have solutions

$$
\begin{aligned}
e^A &= k\frac{\rho^2}{\sinh(2m)} \\
\rho^6 &= \cosh(2m) + \sinh^2(2m)\left(\gamma + \log\left[\frac{\sinh m}{\cosh m}\right]\right).
\end{aligned}
\tag{3.5}
$$

The solution also has non-zero 2-forms [16] but they are zero in the $\theta = \pi/2$ plane which we will analyze below.

Again brane probing has provided the first deep insight into the duality with the field theory. In [19] and [20] it was observed that after substituting the above 10d solution into the DBI action the potential vanishes in the $\theta = \pi/2$ plane. The moduli space for brane motion therefore matches the expected 2d moduli space of the $\mathcal{N} = 2^*$ field theory which has two massless real scalars. From now on we will restrict our attention to this plane. Placing a brane probe off the moduli space corresponds in the field theory to giving a vev to a massive scalar which is neither a vacuum of the theory nor supersymmetric. We know of no field theory results in the presence of such vevs so there are no checks of the duality we can make.

On the moduli space a brane probe reveals the U(1) field theory

$$
\mathcal{L} = \frac{1}{2}\left(\rho^4\cosh(2m)e^{2A}\dot{r}^2 + \frac{L^2\rho^4\cosh(2m)e^{2A}}{\rho^8}\dot{\phi}^2\right) + \frac{1}{4}\tau_3(2\pi\alpha')^2e^{-\Phi}F^{\mu\nu}F_{\mu\nu}\,. \tag{3.6}
$$

In these coordinates the connection to the $\mathcal{N}=2^*$ theory is hidden but we can now find coordinates where the duality is manifest. The two scalar fields should have the same kinetic term with a common coefficient given by the gauge theory's running coupling, $1/g^2_{YM}(r)$. The first of these can be achieved by the change of coordinates

$$v = L\sqrt{\frac{\cosh 2m + 1}{\cosh 2m - 1}} \tag{3.7}$$

such that

$$\frac{\partial v}{\partial r} = \frac{\rho^4}{L} v \tag{3.8}$$

and we have

$$\mathcal{L} = \frac{1}{2}\frac{k^2 L^2 \cosh 2m}{\sinh^2 2m\, v^2}(\dot{v}^2 + v^2\dot{\phi}^2)\,. \tag{3.9}$$

The solutions depend on two constants k and γ which correspond to the mass term and the scalar vev [20] respectively. It is interesting to discuss the anatomy of these solutions at fixed k as a function of γ in the v coordinates. As in previous work [14, 18, 19, 20] we only consider $\gamma \leq 0$ since we can offer no physical interpretation of positive γ. Although, as we will see, $v-\phi$ are not the physical coordinates for the duality they have the benefit of an SO(2) symmetry in ϕ as can be seen from (3.9). The solutions with different choice of the parameter γ differ in the radial position at which the metric has divergences as a result of $\rho \to 0$. From (3.5) and (3.7) one may express γ in terms of this radius l as

$$\gamma = -\frac{l^2}{4L^2} + \frac{L^2}{4l^2} + \ln\frac{l}{L}\,. \tag{3.10}$$

We expect the divergence in the metric to be associated with the presence of a disc D3-brane source and hence solutions with larger negative γ correspond to larger vevs in the field theory. When $\gamma = 0$ the spacetime is good down to a radius $v = l = L$ where $\cosh 2m \to \infty$ and hence the coeffcient of the scalar kinetic term falls to zero. This is the enhançon locus where the probe's tension falls to zero (or in the field theory the coupling diverges) and according to lore [22] we must excise the solution within. Only for this metric can the enhançon point be reached since the other, $\gamma < 0$, solutions have $\rho \to 0$ at a larger radius where the scalar kinetic terms coefficient is still regular.

As pointed out in [20] we can not yet formally make the connection to the gauge coupling because the U(1) theory is not in an $\mathcal{N}=2$ form since the coeffcient of the $F^2_{\mu\nu}$ term is given by

$$e^{-\Phi} = \frac{c}{g_s|\cos\phi + ic\sin\phi|^2}\,. \tag{3.11}$$

To obtain an $\mathcal{N}=2$ form we must make a conformal transformation in the $v-\phi$ plane to equate the coefficients of the scalar and gauge field kinetic terms. The

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transformation is [20]

$$Y = \frac{kL}{2}\left(\frac{V}{L} + \frac{L}{V}\right), \tag{3.12}$$

where $V = ve^{i\phi}, Y = ye^{i\eta}$ are complex parameters on the 2d space. The low energy theory is then of the desired form with

$$\mathcal{L} = \frac{1}{g_{YM}^2(Y)}|\dot{Y}|^2 + \mathrm{Im}\left(\tau(F^{\mu\nu}F_{\mu\nu} + iF^{\mu\nu}\tilde{F}_{\mu\nu})\right), \tag{3.13}$$

with $4\pi/g_{YM}^2(Y) = \mathrm{Im}\,\tau$ where

$$\tau = \frac{i}{g_s}\sqrt{\frac{Y^2}{Y^2 - k^2L^2}}. \tag{3.14}$$

In these coordinates the background takes the form

$$\begin{aligned} ds^2 &= \frac{1}{g_{YM}}\left(H^{-1/2}dx_{//}^2 + H^{1/2}dy^2\right), \\ C_4 &= \frac{g_{YM}^2}{Hg_s}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \\ \tau_3(2\pi\alpha')^2 e^{-\Phi} &= \frac{1}{g_{YM}^2}, \end{aligned} \tag{3.15}$$

with

$$g_{YM}^2 H = \frac{\sinh^4 2m}{k^4\rho^{12}\cosh 2m}. \tag{3.16}$$

All other fields are zero in the $\theta = \pi/2$ plane. In fact the brane probe does not uniquely fix the form of H since it can be rescaled by an arbitrary power of the Yang Mills coupling and still return the same probe theory. Since the coupling in (3.14) does not contain logarithms such a rescaling will not resolve the discrepancies discussed below.

We claim to have identifed the unique coordinates in which in the $\theta = \pi/2$ plane a brane probe correctly matches the expected form for an $\mathcal{N} = 2$ supersymmetric theory. In these physical coordinates we would expect the remainder of the metric to be a parametrization of field theory operators. To see the predictions for these operators we can expand the H function at large radius in these coordinates.

We note that the final transformation in (3.12) is rather strange since the circle $v = L$ is mapped to the real line of length $2kL$ and everything interior is mapped to exterior points to the line in Y space. Thus the V coordinates are a double cover of the Y space. In the v coordinates one can not take a probe through the enhançon so one should exclude the region $v < 1$.

At large y the v coordinate, from (3.12), is given by

$$v = \frac{2y}{k} - \frac{k\cos 2\eta}{2y} + \frac{k^3}{32y^3}(1 - 5\cos 4\eta) + \cdots. \tag{3.17}$$

Thus at large y we find, using (3.5) (3.12) and (3.16)

$$\begin{aligned}
H = & \frac{L^4k^4}{16y^4} + \frac{L^6k^6}{64y^6}\left(-2+2\frac{l^2}{L^2}-\frac{2L^2}{l^2}+8\ln\left(\frac{y}{l}\right)+6\cos(2\eta)\right)+ \\
& + \frac{L^8k^8}{2^8y^8}\left[3\left(1-\frac{l^2}{L^2}+\frac{L^2}{l^2}+4\ln\left(\frac{y}{L}\right)-2\cos 2\eta\right)^2+\right. \\
& \qquad + 2\cos 2\eta\left(-2+2\frac{l^2}{L^2}-\frac{2L^2}{l^2}+8\ln\left(\frac{y}{l}\right)\right)+ \\
& \qquad \left. + \left(3+2\frac{l^2}{L^2}-2\frac{L^2}{l^2}-8\ln\left(\frac{y}{L}\right)-8\cos 2\eta+14\cos 4\eta\right)\right] + \cdots . \quad (3.18)
\end{aligned}$$

Finally we have arrived at the form for the metric we're interested in. The metric on moduli space, when written in the physical coordinates that explicitly display $\mathcal{N}=2$ supersymmetry in the brane probe, has two functions in it. One is the gauge coupling of the theory and the other, H, remains to be interpreted. We can read off the symmetry properties of operators from H using the same prescription as for the $\mathcal{N}=4$ solution; every factor of y carries mass dimension 1 and the η dependence can be interpreted as SO(2) harmonics $\cos n\eta$ with charge n. Thus one would naturally like to interpret the coefficient of $\cos n\eta$, which has U(1) charge n, as the operator $\mathrm{tr}\,\phi^n$ (with ϕ the massless, two component, complex scalar field) and would expect it to be associated with a factor of $y^{(n+4)}$. The charge zero coefficients would correspond to $\mathrm{tr}\,|\phi|^n$ again associated with a factor of y^{n+4}. There are also mixed operators of the form of a product of these two operator types as can be seen from the presence of a $\cos 2\eta$ term at order $1/y^8$. The presence of logarithms though undermines this interpretation. In the $l\to\infty$ limit one would expect the $\mathcal{N}=2^*$ theory to be on the edge of its moduli space and return to looking like the $\mathcal{N}=4$ metric. In fact at large l the leading terms in l do indeed take the form in (2.19) but we can not neglect the $\log y$ terms in this limit which are absent from the $\mathcal{N}=4$ theory. There appears therefore to be UV logarithmic renormalization. Given that there is logarithmic renormalization we can not rule out power like renormalization either which would further confuse the interpretation.

We will make this discrepancy more manifest in the next section where we deduce the D3-brane distributions from the form of the gauge coupling and show that it does not predict the above form for the field theory operators. In the discussion we will suggest a few possible resolutions of the discrepancy.

3.1 D3 distributions

To highlight the discrepancy between field theory expectations and the H function found in the $\mathcal{N}=2^*$ metric we will determine the D3-brane distribution function for spacetimes with different γ assuming the standard one loop renormalized expression for the prepotential governing the IR of the theory. The field theory is reviewed in [20] and the authors followed this logic for the special case $\gamma = 0$, where in Y space the D3-branes are distributed on a line. We extend the analysis to all γ. The prepotential for the $\mathcal{N} = 2^*$ theory is expected to be

$$\mathcal{F} = \frac{i}{8\pi}\left[\sum_{i\neq j}(a_i - a_j)^2 \ln\left(\frac{(a_i - a_j)^2}{\mu^2}\right) - \sum_{i\neq j}(a_i - a_j + m)^2 \ln\left(\frac{(a_i - a_j + m)^2}{\mu^2}\right)\right], \tag{3.19}$$

where a_i are the scalar vev eigenvalues and μ an RG scale. In the supergravity description of the N=2* theory we expect the scalar vevs to be large with respect to the mass term and hence [20]

$$\tau(Y) = \frac{i}{g_s} + \frac{i}{2\pi}\int \sigma \frac{m^2}{(Y-a)^2} d^2a\,, \tag{3.20}$$

where a is a complex 2d integral in Y space and σ the density of vevs/D3-branes. To match with the supergravity we make the identification $m^2 = k^2\pi/L^2$ [20]. Using this ansatz we can determine the distributions that reproduce the supergravity solution expression for τ. In fact this is all but impossible in Y space since there is no spherical symmetry but we know that in V the distributions are circular out to l and cut off inside at $v = L$.

Remarkably, a simple form for the denisity, σ, for each of the solutions, labelled by γ or equivalently l, can then be found by rewriting (3.20) in V space using (3.12) and using

$$\sigma_v v dv d\phi = \sigma_y y dy d\eta\,. \tag{3.21}$$

Expanding the resulting expression as a power series at large y and inserting an expansion in powers of $1/v$ for σ_v one can show to all orders in the expansion that

$$\sigma_v = \frac{1}{\pi(l^2 - L^4/l^2)}\left(1 + \frac{L^4}{v^4} - 2L^2\frac{\cos(2\phi)}{v^2}\right) \tag{3.22}$$

reproduces the supergravity expression (3.14). Note that this result agrees with that of [20] for $\gamma = 0$, $l = L$; integrating with a measure vdv from $v = L$ to l and then taking the $l \to L$ limit we obtain an expression for the number of D3-branes of the form

$$N_{D3} = \frac{1}{\pi}\int_0^\pi (1 - \cos 2\theta) d\theta\,. \tag{3.23}$$

Changing variables to $y = kL\cos\theta$ this reproduces the line density in [20]

$$\sigma_y = \frac{2}{m^2}\sqrt{k^2 - y^2}\,. \tag{3.24}$$

Having identified the density we can then predict the expected scalar operators. Since the only renormlization in the $\mathcal{N} = 2^*$ theory is that of τ [32] we would expect the $\mathcal{N} = 4$ expression for the metric quantity H when evaluated in the $\theta = \pi/2$

plane to display the full set of operators. Thus using (2.16) (with y rescaled to $2y/k$), performing the integration after a change of variables to V space using

$$
\begin{aligned}
y\cos\eta &= \frac{kL}{2}\left(v+\frac{1}{v}\right)\cos\phi\,, \qquad y\sin\eta = \frac{kL}{2}\left(\frac{1}{v}-v\right)\sin\phi \\
y^2 &= \frac{k^2L^2}{4}\left(v^2+\frac{1}{v^2}+2\cos\phi-2\sin\phi\right)
\end{aligned} \tag{3.25}
$$

and further expanding at large y and evaluating the expression in the $\theta=\pi/2$ plane we obtain a prediction for H

$$
\begin{aligned}
H = &\frac{L^4k^4}{16y^4}+\frac{L^6k^6}{64y^6}\left(\frac{2l^2}{L^2}+\frac{2L^2}{l^2}+6\cos(2\eta)\right)+ \\
&+\frac{k^8L^8}{2^8y^8}\left(3\left(\frac{l^4}{L^4}+4+\frac{L^4}{l^4}\right)+16\frac{l^2}{L^4}\left(1+\frac{L^4}{l^4}\right)\cos 2\eta+20\cos 4\eta\right).
\end{aligned} \tag{3.26}
$$

This expression does not match that in (3.18) highlighting the apparent discrepancy in the interpretation of the coefficients as the scalar operators. There appears to be extra logarithmic and power renormalization in the supergravity theory that this simple field theory analysis has not explained.

4. Discussion

Our philosophy has been to try to understand how supergravity solutions dual to field theories encode the field theory operators and their running, concentrating on gravity duals obtained by deforming the 5d supergravity AdS/CFT correspondence. One would expect to be able to interpret all elements of the supergravity solution in this light. In the $\mathcal{N}=4$ theory this is indeed the case with the metric being a simple encoding of the scalar operator vevs. To extend this understanding to theories with more interesting renormlization properties requires undestanding the complicated procedure of including relevant deformations in 5d supergravity, lifting the solutions to 10d and then finding the physical coordinates appropriate to the duality. We have first persued this chain of analysis for the $\mathcal{N}=4$ theory on moduli space with the scalar vevs introduced as deformations. Brane probing has proved to be the vital tool for identifying the physical coordinates; the probe allows us to identify the moduli space and the functional form of the gauge coupling on that moduli space. Matching the gauge coupling to field theory expectations in the $\mathcal{N}=4$ theory provides the physical coordinates in which the encoding of the field theory operators are manifest. This prescription then allows the class of solutions to be extended to describe the full moduli space of the theory.

Armed with this tool we have applied it to the $\mathcal{N}=2^*$ gravity dual. Brane probing the solution reveals the 2d moduli space and, identifying the unique coordinates in which the U(1) theory on the probe takes an $\mathcal{N}=2$ form, the gauge coupling on that moduli space. These should be the physical coordinates in which the duality to the field theory is manifest in the rest of the metric. The metric indeed takes a form on the moduli space analagous to the metric on moduli space in the $\mathcal{N}=4$ theory except that the running of the gauge coupling is also encoded. There is one other function in the metric from which we can read off operators by their scaling dimension and their symmetry properties. In the field theory we expect the gauge coupling to be the only renormalized quantity and the operators $\mathrm{tr}\,\phi^n$ and $\mathrm{tr}\,|\phi|^n$ to emerge as in the $\mathcal{N}=4$ case. In fact we find further renormalization including UV logarithmic renormalization.

The appearance of this extra renormalization is frustrating because it stops us from completely understanding the prescription for creating a gravity dual to a field theory even in the next simplist case to the $\mathcal{N}=4$ theory. The form of the metric on moduli space in (3.15) is highly suggestive that the prescription is to encode the running coupling as shown and then parametrizes the scalar vev operators in the field theory through H. It may be that the discrepancies we have seen are complications brought in by the 5d supergravity approach to constructing the dualities. One possibility is that we have not only introduced a mass term into the field theory. In the $\mathcal{N}=4$ theory when one attempts to introduce a dimension 2 operator at the level of 5d supergravity, after the lift to 10d, a whole host of higher dimension operators are found to be present to make the solution consistent (as can be seen in (2.19)). Something similar may be happening here and the $\mathcal{N}=2^*$ solution is encoding both the field theory scalar vevs and this unknown tower of deformations.

An alternative possibility is that the 5d supergravity solution was created in the coordinates V which are a double cover of the physical coordinates Y. We have excised the solution interior to $v=L$ but possibly there is additional interior structure which in the Y coordinates is projected to large y. Possibly in the physical coordinates there are D3-branes through out the whole space!

In spite of these obstacles we believe the philosophy of the paper, and its support from the $\mathcal{N}=4$ solutions and the simple form of the metric on moduli space in the $\mathcal{N}=2^*$ solutions, will be of use in the investigation and construction of dualities in the future. Our work has hopefully also added to the understanding of the power of brane probing as a tool in such investigations.

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